**Module 1: Review of College Algebra**

**Topics**

1. [Basic Concepts of Graphs](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#I:_Basic_Concepts_of_Graphs)
   1. [Graphing Points and Equations](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#A._Graphing_Points_and_Equations)
   2. [The Pythagorean Theorem, the Distance Formula, and the Midpoint Formula](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#B._The_Pythagorean)
   3. [Completing the Square](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#C._Completing_the_Square)
   4. [Circles](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#Circles)
2. [Basic Concepts of Functions](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#II._Basic_Concepts_of_Functions)
   1. [Functions](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#A._Functions)
   2. [Graphs and Zeros of Functions](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#B._Graphs_and_Zeros_of_Functions)
   3. [The Algebra of Functions](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#C._The_Algebra_of_Functions)
   4. [The Absolute-Value Function](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#D._The_Absolute_Value_Function)
   5. [Polynomials](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#E._Polynomials)
   6. [Rational Functions and Asymptotes](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#F._Rational_Functions_and_Asymptotes)
3. [Symmetry, Transformation of Functions, and Inverse Functions](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#III._Symmetry)
   1. [Symmetry](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#A._Symmetry)
   2. [Transformations of Functions](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#B._Transformations_of_Functions)
   3. [Inverse Functions](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#C._Inverse_Functions)
4. [Solutions of Equations and Inequalities](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#IV:_Solutions_of_Equations_and_Inequalities)
   1. [Solution of an Equation](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#A._Solution)
   2. [Linear Equations](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#B._Linear)
   3. [Complex Numbers](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#C._Complex_Numbers)
   4. [Quadratic Equations](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#D._Quadratic_Equations)
   5. [Rational Equations](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#E._Rational_Equations)
   6. [Solution of Inequalities](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#F._Solution_of_Inequalities)

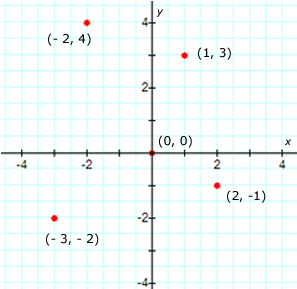
**I. Basic Concepts of Graphs**

After completing this section, you should be able to:

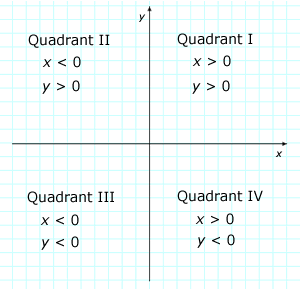
* graph ordered pairs and equations in two variables
* determine if an ordered pair is a solution of a given equation
* identify intercepts of graphs
* state the Pythagorean theorem
* calculate the distance between two points in a plane
* calculate the midpoint between two points in a plane
* carry out the technique of completing the square
* graph a circle and write the equation of a circle in standard form

**A. Graphing Points and Equations**

Points in the plane are plotted using the Cartesian coordinate system. The horizontal *x*-axis and the vertical *y*-axis intersect at the origin (0, 0). Each point in the plane is represented by an ordered pair (*x*, *y*).

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The *x*-axis and the *y*-axis divide the plane into four quadrants.

****

**Solution of an Equation**

An equation involving the variables *x* and *y* defines a relationship between *x* and *y*.

A *solution* of the equation is an ordered pair which makes the equation true.

**Example I.A.1:** Consider the equation 3*x* – 2*y* = 24.

(4, –6) is a solution because 3(4) – 2(–6) = 12 + 12 = 24.

However, (6, 3) is not a solution, because 3(6) – 2(3) = 18 – 6 = 12 ≠ 24.

A *graph* of an equation is a plot of all the solutions of the equation.

Certain points on a graph are of particular interest.

**Intercepts**

An *x*-*intercept* is a point where a graph intersects the *x*-axis.   
It is a point of the form (*a*, 0) on a graph.   
To find *a*, let *y* = 0 and solve for *x*.

A *y*-*intercept* is a point where a graph intersects the *y*-axis.  
It is a point of the of the form (0, *b*) on a graph.  
To find *b*, let *x* = 0 and solve for *y*.

**Example I.A.2:** Find the intercepts of the graph of the equation 3*x* – 2*y* = 24.

**Solution:**

|  |  |
| --- | --- |
| To find the *x*-intercept, let *y* = 0. Then solve for *x*:             3*x* – 2(0) = 24                      3*x* = 24                        *x* = 24/3                        *x* = 8  The *x*-intercept is (8, 0).  To find the *y*-intercept, let *x* = 0. Then solve for *y*:             3(0) – 2*y* = 24                    –2*y* = 24                        *y* = 24/(–2)                        *y* = –12  The *y*-intercept is (0, –12). |  |

**B. The Pythagorean Theorem, the Distance Formula, and the Midpoint Formula**

**Pythagorean Theorem**

|  |  |
| --- | --- |
| For any right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.  If the lengths of the legs are *a* and *b*, and the length of the hypotenuse is *c*, then  *a*2 + *b*2 = *c*2 |  |

The Pythagorean theorem can be applied to find the distance between two points in the plane.

**Distance Formula**

|  |  |
| --- | --- |
| The distance *d* between any two points (*x*1, *y*1) and (*x*2, *y*2) is given by |  |

Given the points (*x*1, *y*1) and (*x*2, *y*2), a right triangle can be drawn, as shown above. The length of the hypotenuse, *d*, is the distance between the points (*x*1, *y*1) and (*x*2, *y*2).

Since the points (*x*1, *y*1) and (*x*2, *y*1) lie on a horizontal line, the distance between them is |*x*2 – *x*1|. (Recall the connection between [absolute value and distance](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/sa9.1-8-0orig.gif/absolutevalue.html).)

Since the points (*x*2, *y*1) and (*x*2, *y*2) lie on a vertical line, the distance between them is  
|*y*2 – *y*1|. Therefore, the lengths of the legs of the triangle are |*x*2 – *x*1| and |*y*2 – *y*1|.

Applying the Pythagorean theorem, *d*2 = |*x*2 – *x*1|2 + |*y*2 – *y*1|2.

Since the square of any real value is nonnegative, |*x*2 – *x*1|2 = (*x*2 – *x*1)2  
and |*y*2 – *y*1|2 = (*y*2 – *y*1)2.

So, *d*2 = (*x*2 – *x*1)2 + (*y*2 – *y*1)2.

Taking the square root, .

**Example I.B.1:** Find the distance between the points (–3, 5) and (2, 7).

Solution:



**Midpoint Formula**

|  |  |
| --- | --- |
| If the endpoints of a line segment are (*x*1, *y*1) and (*x*2, *y*2) then the coordinates of the midpoint are | midpoint graph |

**Example I.B.2:** Find the midpoint of the segment having endpoints (–3, 5) and (2, 7).

**Solution:**



**C. Completing the Square**

In order to graph equations of circles and parabolas, and to solve quadratic equations, it is often helpful to apply the algebraic technique called *completing the square.* This method will be used in [topic I-D](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#Circles), [topic II-E](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#E._Polynomials), and [topic IV-D](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#D._Quadratic_Equations).

The technique of completing the square is applied to an expression of the form  
*ax*2 + *bx* + *c*, where *a, b,* and *c* are real-valued constants.

Although the expression *ax*2 + *bx* + *c* may not necessarily be factorable, it is always possible to write the expression in the form *a*(*x* + *r*)2 + *s*.

The technique is simplest for the expression *x*2 + *bx* + *c*, where the leading coefficient, *a*, is equal to 1.

The idea is to figure out what "magic number" to add to *x*2 + *bx* in order to arrive at the square of a binomial.

Take half of the coefficient of the *x* term, ½*b*, and square it to get ¼*b*2.

Then for the original expression, both add ¼*b*2 and subtract ¼*b*2.

|  |  |  |
| --- | --- | --- |
| So, *x*2 + *bx* + *c* | =   *x*2 + *bx* + (¼*b*2 – ¼*b*2) + *c* | Add and subtract ¼*b*2. |
|  | =   (*x*2 + *bx* + ¼*b*2) – ¼*b*2 + *c* | Regroup. |
|  | =   (*x* + ½*b*)2 + (*c*– ¼*b*2) | Factor and regroup. |

**Example I.C.1:** Complete the square for the trinomial *x*2 – 6*x* + 7.

**Solution:** Since the leading coefficient is 1 and the coefficient of the *x* term is 6, take ½(–6) = –3, and square –3 to get (–3)2 = 9.

|  |  |  |
| --- | --- | --- |
| *x*2 – 6*x* + 7 | = *x*2 – 6*x* + 9 – 9 + 7 | Add and subtract 9. |
|  | = (*x*2 – 6*x* + 9) – 9 + 7 | Regroup. |
|  | = (*x* – 3)2 – 2 | Factor. |

For a more general expression of the form *ax*2 + *bx* + *c*, first factor out *a* and then apply the procedure described above.

**Example I.C.2:** Complete the square for the trinomial 2*x*2 + 16*x* + 40.

**Solution:** Since the leading coefficient is 2, start by factoring it out of the *x* terms.

|  |  |  |
| --- | --- | --- |
| 2*x*2 + 16*x* + 40 | = 2(*x*2 + 8*x*) + 40 | Factor out 2 from the *x* terms. |
|  | = 2(*x*2 + 8*x* + 16 – 16) + 40 | Add and subtract [½(8)]2 = 42 =16. |
|  | = 2(*x*2 + 8*x* + 16) – 32 + 40 | Multiply 2(–16) and regroup. |
|  | = 2(*x* + 4)2 + 8 | Factor and simplify the constant. |

In the following topic, the technique of completing the square will be applied to a general equation of a circle, in order to find the center and the radius.

**D. Circles**

A circle is determined by its center point (*h*, *k*) and its radius *r*. A circle is the set of all points which are located a given distance *r* from the center (*h*, *k*).

**Equation of a Circle**

|  |  |
| --- | --- |
| The equation of a circle with center (*h*, *k*) and radius *r*, in standard form, is  (*x* – *h*)2 + (*y* – *k*)2 = *r*2 |  |

**Example I.D.1:** Graph the circle whose equation is *x*2 + *y*2 = 1.

Solution:

|  |  |
| --- | --- |
| *x*2 + *y*2 = 1 can be written as  (*x* – 0)2 + (*y* – 0)2 = 12  This equation represents the circle of radius 1 centered at (0, 0), the origin. This particular circle is known as the *unit circle*. |  |

More generally, an equation of a circle may be written as *x*2 + *y*2 + *bx* + *cy* + *d* = 0. In this form, the center and radius are not immediately apparent. However, by completing the square, the equation can be transformed to standard form.

**Example I.D.2:** Find the center and the radius of the circle given by the general equation

*x*2 + *y*2 – 6*x* + 10*y* – 2 = 0.

**Solution:**

|  |  |
| --- | --- |
| *x*2 + *y*2 – 6*x* + 10*y* – 2 = 0 |  |
| (*x*2 – 6*x*) + (*y*2 + 10*y*) – 2 = 0 | Group terms. |
| (*x*2 – 6*x* + 9 – 9) + (y2 + 10*y* + 25 – 25) – 2 = 0 | ½(–6) = –3, and (–3)2 = 9. ½(10) = 5, and 52 = 25. |
| (*x*2 – 6*x* + 9) + (*y*2 + 10*y* + 25) – 9 – 25 – 2 = 0 | Regroup. |
| (*x* – 3)2 + (*y* + 5)2 – 36 = 0 | Factor and simplify. |
| (*x* – 3)2 + [*y* – (–5)]2 = 62 | Write in standard form. |
| The center is (3, –5) and the radius is 6. | |

**II. Basic Concepts of Functions**

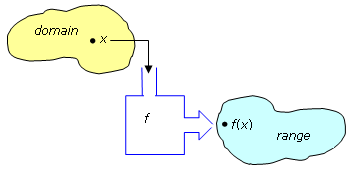
After completing this section, you should be able to:

* identify functional relationships
* determine the domain of a function
* find the sum, difference, product, quotient, and composition of two functions
* analyze graphs of polynomials and determine the behavior at the ends as |*x*|→∞
* graph linear, quadratic, and rational functions
* find the equation of a line
* determine the vertex of a quadratic function
* determine the domain and asymptotes of a rational function

**A. Functions**

**Function**

A **function** defined on a set (called the *domain*) is a rule that assigns to each value *x* in the domain exactly one value *f*(*x*) in a second set (called the *range*).



Unless stated otherwise, assume that the domain and range of a given function are subsets of the real numbers.

For a function of the form *y* = *f*(*x*), the domain is the set of all real-number inputs *x* that result in a real-number output *y.* The range is the set of outputs *y*.

**Example II.A.1:** Find the domain of the function .

**Solution:**

For *x* = 1, *f*(*x*) = *f*(1) = –1. For *x* = 0, *f*(*x*) = *f*(0) = 0. However, *f*(2) = 2/0, which is undefined. Therefore, 2 is not in the domain of the function. All other real inputs do result in real-number outputs.

The domain of *f* consists of all real numbers except 2;  
thus, the domain of *f* = {*x* | *x* ≠ 2} = (–∞, 2) U (2, ∞).

In general, to determine the domain of a function involving a rational expression, exclude any values which result in a zero denominator.

**Example II.A.2:** Find the domain of the function .

**Solution:**

In order to get a real-valued output, the quantity under the square-root sign, *x* – 5, must be nonnegative. Therefore, it must be true that *x* – 5 ≥ 0, and so *x* ≥ 5.

The domain is {*x* | *x* ≥ 5} = [5, ∞).

In general, when finding the domain of a function involving a radical, exclude any values for which the radical does not exist as a real number.

**B. Graphs and Zeros of Functions**

In order to be a function, no *x*-value can be assigned more than one *y*-value. If any vertical line crosses the graph more than once, then there is an *x*-value which has more than one corresponding *y*-value, and the graph does not represent a function.

**Vertical-Line Test**

If it is possible to draw a vertical line that crosses a graph more than one time, then the graph does not represent a function.

|  |  |
| --- | --- |
|  |  |
| Since a vertical line intersects the graph more than one time, this graph does not represent a function. | Since no vertical line intersects the graph more than once, this graph does represent a function. |

**Zero of a Function**

**Zero of a Function**

A number *c* for which *f*(*c*) = 0 is called a *zero* of the function *f*.

Example II.B.1:

|  |  |
| --- | --- |
| The graph of *f*(*x*) = *x*2 – 3*x* + 2 is shown to the right.  Since *f*(1) = 0 and *f*(2) = 0, 1 and 2 are zeros of *f*. |  |

Recall that an *x*-intercept is a point of the form (*c*, 0) where a graph intersects the *x*-axis. For a function *y* = *f*(*x*), if (*c*, 0) is on the graph, then *y* = *f*(*c*) = 0 and *c* is a zero.

Finding the zeros of a function *f* entails finding the solutions of the equation *f*(*x*) = 0.

The number of zeros depends upon the function. The function in the example above has two zeros. Some functions have no real-valued zeros. Some functions have a finite number of zeros. In module 2, you will study some functions that have infinitely many zeros.

The process of solving an equation is discussed in [topic IV](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#IV:_Solutions_of_Equations_and_Inequalities) of this module.

**C. The Algebra of Functions**

Two functions may be combined to form new functions, using the operations of addition, subtraction, multiplication, division, and composition. The following table summarizes the types of combinations and provides examples.

|  |  |  |
| --- | --- | --- |
| **Type of Function** | **Definition** | **Example: *f*(*x*) = 3*x*2 + 2*x* + 1 *g*(*x*) = 5*x* – 4** |
| Sum | (*f* + *g*)(*x*) = *f*(*x*) + *g*(*x*) where *x* is in the domains of both *f* and *g* | (*f* + *g*)(*x*)    = (3*x*2 + 2*x* + 1) + (5*x* – 4)    = 3*x*2 + 7*x* – 3 |
| Difference | (*f* – *g*)(*x*) = *f*(*x*) – *g*(*x*) where *x* is in the domains of both *f* and *g* | (*f* – *g*)(*x*)     = (3*x*2 + 2*x* + 1) – (5*x* – 4)     = 3*x*2 – 3*x* + 5 |
| Product | (*fg*)(*x*) = *f*(*x*) *g*(*x*) where *x* is in the domains of both *f* and *g* | (*f* *g*)(*x*)     = (3*x*2 + 2*x* + 1)(5*x* – 4)     = 15*x*3 – 2*x*2 – 3*x* – 4 |
| Quotient | (*f*/*g*)(*x*) = *f*(*x*)/*g*(*x*),  where *x* is in the domains of both *f* and *g*, and *g* (*x*) ≠ 0 | , *x* ≠ 4/5 |
| Composition | (*f ◦ g*)(*x*) = *f*(*g*(*x*)) where *x* is in the domain of *g* and *g*(*x*) is in the domain of *f* | (*f* *◦ g*)(*x*) = *f*(5*x* – 4)     = 3(5*x* – 4)2 + 2(5*x* – 4) + 1     = 3(25*x*2 – 40*x* + 16) + 10*x* – 8 + 1    = 75*x*2 – 110*x* + 41 |

There are certain types of functions that are often encountered, including the absolute-value function, polynomial functions, and rational functions. The rest of topic II is devoted to reviewing these types of functions.

**D. The Absolute-Value Function**

The **absolute value** of *x*, denoted |*x*|, is the distance between *x* and 0 on the number line.

Note that distance is always a nonnegative number. For example, the distance between 4 and 0 is |4| = 4. The distance between –4 and 0 is |–4| = 4.

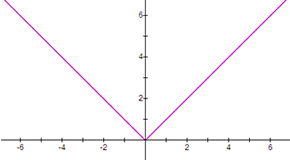
The absolute value of *x* can also be expressed in terms of a formula.

Absolute value: 

**Absolute-Value Function**

The **absolute-value function** is given by *f*(*x*) = |*x*|.

The graph of the absolute-value function is shown below.



The domain of the absolute-value function consists of all real numbers and the range consists of all nonnegative numbers. The graph of the absolute-value function has a distinctive V-shape.

**E. Polynomials**

A **polynomial function** has the form

*P*(*x*) = *an xn*+ *an*– 1*xn*– 1 + *an*– 2 *xn*– 2+ … + *a*1*x* + *a*0,

where the coefficients *an*, *an*– 1, …, *a*1, and *a*0 are real numbers, *an* ≠ 0, and the exponents are positive whole numbers. The degree of the polynomial is *n*, the largest exponent; the leading coefficient is *an*, the coefficient associated with *xn*; and the leading term is *an xn*.

For example, *P*(*x*) = 5*x*4 – 6*x*3 – 8*x*2 + *x* – 1 is a polynomial of degree 4, the leading coefficient is 5, and the leading term is 5*x*4.

The polynomial *P*(*x*) = 0 is called the *zero polynomial,* and by definition, it has no degree. The graph of the zero polynomial is the *x*-axis, *y* = 0.

The graphs of polynomials of low degree have distinctive shapes.

|  |  |  |  |
| --- | --- | --- | --- |
| **Degree and Type** | **Polynomial and Type of Graph** | **Degree and Type** | **Polynomial and Type of Graph** |
| 0 Constant Linear | *P*(*x*) = *a*0, with *a*0 ≠ 0  Horizontal Line | 1  First- Degree  Linear | *P*(*x*) = *a*1*x* + *a*0, with *a*1 ≠ 0  Line  The line is neither horizontal nor vertical. |
| 2 Quadratic | *P*(*x*) = *a*2*x*2 +*a*1*x* + *a*0, with *a*2 ≠ 0  Parabola | 3 Cubic | *P*(*x*) = *a*3*x*3 + *a*2*x*2 +*a*1*x* + *a*0, with *a*3 ≠ 0  Cubic |

For polynomials of higher degree, it is not usually so easy to determine the graph. However, it is possible to easily determine the behavior of the graphs at the ends, as |*x*|→∞, by examining the leading term.

|  |  |  |  |
| --- | --- | --- | --- |
| **Degree of Polynomial, *n*** | **Sign of Leading Coefficient *an*** | **Behavior of Graph** | |
| Even | Positive |  | Upward to the left and upward to the right |
| Even | Negative |  | Downward to the left and downward to the right |
| Odd | Positive |  | Downward to the left and upward to the right |
| Odd | Negative |  | Upward to the left and downward to the right |

**Example II.E.1:** Determine the behavior of the graph of

|  |  |
| --- | --- |
| *P*(*x*) = 5*x*4 – 6*x*3 – 8*x*2+ *x* – 1 as |*x*| → ∞.  Solution:  Since the degree, 4, is even, and the leading coefficient, 5, is positive, the graph goes upward to the left (as *x* → –∞) and upward to the right (as *x* → ∞). |  |

The remainder of this section will concentrate on linear functions (polynomials of degree 0 and 1) and quadratic functions (polynomials of degree 2).

**Linear Functions**

**Linear Function**

A **linear function** has the form *f*(*x*) = *mx* + *b*, where *m* and *b* are constants.

The graph of a linear function is a straight line. The coefficient *m* measures the slope, the steepness of the line.

**Slope**

|  |  |
| --- | --- |
| The *slope*of a line containing points (*x*1, *y*1) and (*x*2, *y*2), where *x*1≠ *x*2, is given by |  |

If the change in *x* is zero, the slope is undefined; otherwise, the slope is a real number. If the change in *x* and the change in *y* have the same sign, the slope is positive. If the change in *x* and the change in *y* have opposite signs, the slope is negative. If the change in *x* is nonzero and the change in *y* is zero, then the slope is zero. These cases are explored in the discussion below.

**Example II.E.2:** Find the slope of the line containing the points (1, 8) and (4, –1).

Solution:

|  |  |
| --- | --- |
|  |  |

From the graph of a line, it is easy to tell whether the slope is positive, negative, 0, or undefined.

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
| Positive slope  Change in *x* and change in *y* have the same sign. | Negative slope  Change in *x* and change in *y* have the opposite signs. | The slope of a horizontal line is 0.  Change in *y* is 0 (*y* remains constant). | The slope of a vertical line is undefined.  Change in *x* is 0 (*x* remains constant). |

A horizontal line has an equation of the form *y* = *b*, where *b* is a constant.

A vertical line has an equation of the form *x* = *a*, where *a* is a constant.

In general, an equation of a nonvertical line may be written in several forms.

**Slope-Intercept Form**

*y* = *mx* + *b*, where *m* is the slope and (0, *b*) is the *y*-intercept

**Point-Slope Form**

*y* – *y*1 = *m*(*x* – *x*1), where *m* is the slope and (*x*1, *y*1) is a point on the line

**Example II.E.3:** Find an equation of the line containing the points (1, 8) and (4, –1).

**Solution:**

|  |  |
| --- | --- |
| In the previous example, the slope was determined to be –3.  Using *m* = –3 and the point (1, 8), an equation of the line is:  *y* – 8 = –3(*x* – 1)    Point-slope form. *y* – 8 = –3*x* + 3 *y* = –3*x* + 11    Slope-intercept form.  Note that the *y*-intercept is (0, 11). |  |

**Parallel Lines and Perpendicular Lines**

|  |  |  |  |
| --- | --- | --- | --- |
| Vertical lines are parallel.  Nonvertical **parallel** lines have the same slope. | Parallel lines | If one line is vertical and another is horizontal, they are perpendicular.  If one of the lines is not vertical, then two lines are **perpendicular** if the product of their slopes is –1. | Perpendicular lines |

**Example II.E.4:** Find an equation of the line perpendicular to the line *y* = –3 x + 11 and containing the point (2, 5).

Solution:

|  |  |
| --- | --- |
| The line *y* = –3*x* + 11 has slope of –3.  The slope of the perpendicular line, *m*, must satisfy –3*m* = –1, so *m* = 1/3.  Using *m* = 1/3 and the point (2, 5), an equation of the perpendicular line is:  *y* – 5 = 1/3(*x* – 2)         Point-slope form. *y* – 5 = 1/3*x* – 2/3       y = 1/3*x* + 13/3     Slope-intercept form. |  |

**Quadratic Functions**

**Quadratic Function**

A **quadratic function** has the form *f*(*x*) = *ax*2 + *bx* + *c*, where *a*, *b*, and *c* are real numbers, and *a* ≠ 0. The graph of a quadratic function is called a *parabola*.

|  |  |
| --- | --- |
| If the leading coefficient, *a*, is positive, then the parabola opens upward: | Quadratic Function |
| If the leading coefficient,*a*, is negative, then the parabola opens downward: | Quadratic Function |

The point at which the graph changes direction is the vertex.  The vertex can be found by applying the technique of completing the square (reviewed in Topic I-C).

|  |  |
| --- | --- |
| *f*(*x*) = *ax*2 + *bx* + *c* can be rewritten in the form  *f*(*x*) = *a*(*x* – *h*)2 + *k*  where and .  The point (*h*, *k*) is the vertex, and the vertical line *x* = *h* is the **axis of symmetry**, the line that bisects the parabola into two mirror images. |  |

**Example II.E.5:** Find the vertex and axis of symmetry of the graph of the function

*f*(*x*) = –*x*2 + 10*x* – 27.

**Solution:**

Completing the square:

|  |  |  |
| --- | --- | --- |
| *f*(*x*) | = –*x*2 + 10*x* – 27 |  |
|  | = –(*x*2 – 10*x*) – 27 | Factor out –1 from the *x* terms. |
|  | = –(*x*2 – 10*x* + 25 – 25) – 27 | Add and subtract [½(–10)]2 = (–5)2 = 25. |
|  | = –(*x*2 – 10*x* + 25) + 25 – 27 | Multiply –(–25) and regroup. |
|  | = –(*x* – 5)2 – 2 | Factor and simplify the constant. |
| Vertex: (*h*, *k*) = (5, –2).  The axis of symmetry is the line *x* = 5.  **Alternate Method:**  Applying the formulas for the coordinates of the vertex,  *h* = –*b*/(2*a*) = –10/[2(–1)] = 5  *k* = *f*(5) = –(5)2 + 10(5) – 27 = –2.  The axis of symmetry is the line *x* = 5. | |  |

The properties of quadratic functions and their graph are summarized below.

|  |  |  |
| --- | --- | --- |
| Quadratic Function: *f*(*x*) = *a*(*x* – *h*)2 + *k*  Vertex: (*h*, *k*) Axis of symmetry: vertical line *x* = *h* | | |
| If *a* > 0, the parabola opens upward,  the minimum value is *k*, and the range is [*k*, ∞). |  | If *a* < 0, the parabola opens downward,  the maximum value is *k*, and  the range is (–∞, *k*]. |
|  |  |  |

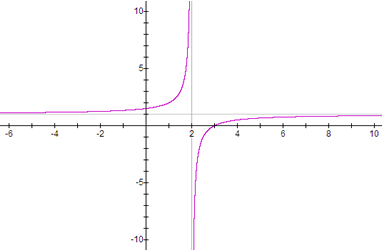
**F. Rational Functions and Asymptotes**

**Rational Function**

A **rational function** is a quotient of two polynomials, , where *p*(*x*) and *q*(*x*) are polynomials and *q*(*x*) is not the zero polynomial.

As noted in [topic II-A](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#A._Functions), inputs for a rational expression must consist only of *x*-values for which the denominator is nonzero. Therefore, the domain of the rational function Rational Function

For example, consider the function  and its graph:



The domain of *f* is {*x* | *x* ≠ 2}. The numerator of *f* is equal to 0 for *x* = 3, and so the graph has an *x*-intercept of (3, 0). The *y*-intercept is (0, 3/2).

The vertical line *x* = 2 is called a *vertical asymptote* for the graph of *f*. As *x* gets closer and closer to 2 from the left, the *y*-values get larger and larger, increasing without bound. As *x* gets closer and closer to 2 from the right, the *y*-values decrease without bound. Notice that the vertical asymptote corresponds to the *x*-value, 2, for which the denominator *x* – 2 is 0.

The horizontal line *y* = 1 is called a *horizontal asymptote* for the graph of *f*. As *x* becomes larger and larger, increasing without bound, the *y*-values get closer and closer to 1. Also, as *x* decreases without bound, the *y*-values get closer and closer to 1.

Note that the asymptotes are not part of the graph of the function. They merely aid in visualizing the trends in the *y*-values of the function.

In general, asymptotes for rational functions can be determined in a systematic way.

**Vertical Asymptotes**

Assuming that a rational function has been written in lowest terms (so *p*(*x*) and *q*(*x*) have no common factors), vertical asymptotes occur at the *x*-values for which the denominator *q*(*x*) is zero.

That is, the line *x* = *a* is a vertical asymptote if and only if *q*(*a*) = 0.

**Example II.F.1:**

Determine the vertical asymptotes for the function .

Solution:

. The denominator (*x* + 3)(*x* – 3) = 0 for *x* = –3 or *x* = 3.

The vertical asymptotes are the vertical lines *x* = –3 and *x* = 3.

For a rational function, the existence of a horizontal asymptote depends on the relative size of the degrees of the polynomials in the numerator and denominator.

**Horizontal Asymptotes**

For a rational function :

If the degree of *p*(*x*) = the degree of *q*(*x*),

the line *y* = *c* is a horizontal asymptote, where .

If the degree of *p*(*x*) < the degree of *q*(*x*), the line *y* = 0 (the *x*-axis) is a horizontal asymptote.

If the degree of *p*(*x*) > the degree of *q*(*x*), there is no horizontal asymptote.

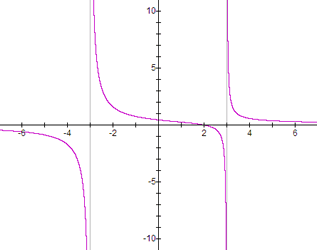
**Example II.F.2:**

Determine the horizontal asymptote (if any) for the function .

**Solution:**

The numerator is a polynomial of degree 1, and the denominator is a polynomial of degree 2. Since the degree of the numerator is less than the degree of the denominator, the line *y* = 0 is the horizontal asymptote.

The graph of  is shown below.



As determined in the previous examples, the function *g* has vertical asymptotes *x* = –3 and *x* = 3, and the horizontal asymptote *y* = 0.

In addition, since the numerator 2*x* – 4 = 0 for *x* = 2, the graph has an *x*-intercept (2, 0). Since *g*(0) = 4/9, the *y*-intercept is (0, 4/9). By plotting some additional points, the general shape can be determined. For example, , so  is a point on the graph.

**III. Symmetry, Transformation of Functions, and Inverse Functions**

After completing this section, you should be able to:

* assess the symmetry of a graph with respect to the *x*-axis, the *y*-axis, and the origin
* determine whether a function is even, odd, or neither
* graph a function obtained by shifting, reflecting, stretching, or shrinking another graph
* determine whether a function is one-to-one and has an inverse

**A. Symmetry**

Graphs may exhibit certain types of symmetry. It is often useful to identify whether a graph is symmetric with respect to the *x*-axis, the *y*-axis, or the origin.

|  |  |  |
| --- | --- | --- |
| **Type of Symmetry** |  | **Symmetry Condition** |
| Symmetry with respect to the *x*-axis | Symmentry | Replacing *y* with –*y* produces an equivalent  equation. |
| Symmetry with respect to  the *y*-axis | Symmetry | Replacing *x* with –*x* produces an equivalent equation. |
| Symmetry with respect to the *origin* | Symmetry | Replacing *x* with –*x* and *y* with –*y* produces an equivalent equation. |

**Example III.A.1:** Determine whether the graph of the equation *y* = *x*3 – 4*x* is symmetric with respect to the *x*-axis, *y*-axis, and/or the origin.

Solution:

The concept of symmetry can also be applied to functions. A function may be categorized as even, odd, or neither even nor odd.

|  |  |  |
| --- | --- | --- |
| **Type of Function** | **Symmetry Condition** | **Property of Graph** |
| Even function | *f*(–*x*) = *f*(*x*) for all *x* in the domain | Graph is symmetric with respect to *y*-axis |
| Odd function | *f*(–*x*) = –*f*(*x*)  for all *x* in the domain | Graph is symmetric with respect to origin |

**Example III.A.2:** Determine whether the function *f*(*x*) = *x*2 + |*x*| is even, odd, or neither even nor odd.

Solution:

*f*(–*x*) = (–*x*)2 + |–*x*| = *x*2 + |*x*| = *f*(*x*), so *f* is an even function.

**Example III.A.3:** Determine whether the function *f*(*x*) = *x*3 + 2 is even, odd, or neither even nor odd.

Solution:

*f*(–*x*) = (–*x*)3 + 2 = –*x*3 + 2  
*f*(–*x*) ≠ *f*(*x*), so *f* is not an even function.

–*f*(*x*) = –*x*3 – 2  
*f*(–*x*) ≠ –*f*(*x*), so *f* is not an odd function.

The function *f* is neither even nor odd.

**B. Transformations of Functions**

Basic functions such as *f*(*x*) = *x*2 can be thought of as building blocks for other functions.

For example, the graph of *g*(*x*) = (*x* – 2)2 + 1 has the same parabola shape as the graph of *f*, but its vertex is located at (2, 1). (Recall that quadratic functions were reviewed in [topic II-E](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#II.E.Quadratic).)

|  |  |
| --- | --- |
| To arrive at the parabola  *y* = (*x* – 2)2 + 1, shift the parabola *y* = *x*2 to the right by 2 units and upward by 1 unit.  The vertex has been shifted from (0, 0) to (2, 1).  Notice that  *g*(*x*) = (*x* – 2)2 + 1 = *f*(*x* – 2) + 1, so the graph of *g* involves a shift of the graph of *f* to the right by 2 units and upward by 1 unit. |  |

This example has illustrated both a horizontal and a vertical translation. Other types of transformations include reflection, vertical stretching and shrinking, and horizontal stretching and shrinking. The table below summarizes these types of transformations.

|  |  |
| --- | --- |
| **Transformations of Functions** |  |
| Vertical translation: *y* = *f*(*x*) ± *b*, for *b* > 0 | |
| The graph of *y* = *f*(*x*) + *b* is the graph of *y* = *f*(*x*) shifted upward *b* units. |  |
| The graph of *y* = *f*(*x*) –*b* is the graph of *y* = *f*(*x*) shifted downward *b* units. |  |
| Horizontal translation: *y* = *f*(*x* https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/images/mod1hor.gif*b*),  for *b* > 0 | |
| The graph of *y* = *f*(*x*–*b*) is the graph of *y* = *f*(*x*) shifted rightward *b* units. |  |
| The graph of *y* = *f*(*x*+ *b*) is the graph of *y* = *f*(*x*) shifted leftward *b*units. |  |
| Reflection: *y* = –*f*(*x*) or *y* = *f*(–*x*) | |
| The graph of *y* = –*f*(*x*) is the reflection of the graph of *y* = *f*(*x*) across the *x*-axis. |  |
| The graph of *y* = *f*(–*x*) is the reflection of the graph of *y* = *f*(*x*) across the *y*-axis. |  |
| Vertical Stretching or Shrinking: *y* = *cf*(*x*) | |
| For |*c*| > 1, the graph of *y* = *cf*(*x*) is a vertical stretching of the graph of *y* = *f*(*x*). |  |
| For 0 < |*c*| < 1, the graph of *y* = *cf*(*x*) is a vertical shrinking of the graph of *y* = *f*(*x*). |  |
| Horizontal Stretching or Shrinking: *y* = *f*(*cx*) | |
| For |*c*| > 1,  the graph of *y* = *f*(*cx*) is a horizontal shrinking of the graph of *y* = *f*(*x*). |  |
| For 0 < |*c*| < 1, the graph of *y* = *f*(*cx*) is a horizontal stretching of the graph of *y* = *f*(*x*). |  |

**Example III.B.1:** Describe how the graph of *g*(*x*) = –(*x* + 4)2 – 1 can be obtained from the graph of *f*(*x*) = *x*2.

Solution:

Since *f*(*x* + 4) = (*x* + 4)2, the function *g* has the form

*g*(*x*) = –(*x* + 4)2 – 1 = *–f*(*x* + 4) – 1.

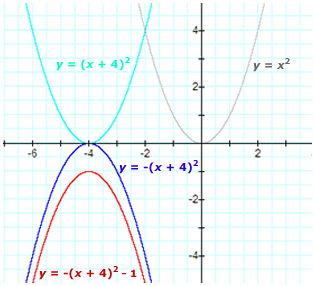
Follow the usual order of operations.

First consider *f*(*x* + 4), then multiplication by –1, and then the final subtraction.

The graph of *y* = *f*(*x* + 4) is a shift of *f*(*x*) = *x*2 to the left by 4 units.

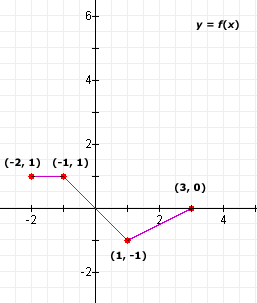
The graph of *y* = –*f*(*x* + 4) is a shift to the left by 4 units, and a reflection across the *x*-axis.

The graph of *y* = –*f*(*x* + 4) – 1 is a shift to the left by 4 units, a reflection across the *x*-axis, and then a shift downward by 1 unit.



**Example III.B.2:**

Given the graph of *y* = *f*(*x*) below, sketch the graph of *y* = –2 *f*(*x* – 1) + 3.



|  |  |
| --- | --- |
| Solution: | |
| **Step 1.**  Shift the graph of *y* = *f*(*x*)  by 1 unit to the right  to arrive at the graph of *y* = *f*(*x* – 1). | **Step 2.**  Stretch the graph of *y* = *f*(*x* – 1) by a factor of 2 vertically to arrive at the graph of *y* = 2 *f*(*x* – 1). |
|  | *f*–1" for the inverse does not denote the reciprocal of *f*. The domain of the inverse function *f*–1 is the range of *f* and the range of the inverse function *f*–1 is the domain of *f*.  For example, suppose *f*(*x*) = 3*x* – 4. To find the inverse:  *y* = 3*x* – 4     Replace *f*(*x*) with *y.*  *y* + 4 = 3*x*     Simplify.        Solve for *x.*  Now *x* has been written as a function of *y*, and the inverse function can be expressed as   Typically, functions are written with the input variable denoted by *x*, so by convention, the inverse function is written .  (Note that  is not 1/*f*(*x*) = 1/(3*x* – 4).)  The graph of *f* and its inverse are shown below. Note the symmetry about the line *y* = *x*.    If a function is one-to-one, then the inverse function must also be one-to-one. The inverse of the inverse function is the original function.   |  |  | | --- | --- | |  | If *f* has an inverse, and *f*(*r*) = *s*, then *f*–1(*s*) = *r*.  So, (*f*–1 ◦ *f*)(*r*) = *f*–1 (*f*(*r*)) = *f*–1 (*s*) = *r.*  Thus, given an input *r* in the domain of *f*, (*f*–1 ◦ *f*)(*r*) = *r*.  Similarly, (*f* ◦ *f*–1)(*s*) = *f*(*f*–1 (*s*)) = *f*(*r*) = *s*.  Thus, given an input *s* in the domain of *f*–1, (*f* ◦ *f*–1)(*s*) = *s*. |   Below is a summary of the properties of a one-to-one function.  **Properties of a One-to-One Function *f***  Different inputs have different outputs. No two inputs have the same output.  *f* has an inverse function, denoted *f*–1.  The domain of *f* is the range of the inverse *f*–1.  The range of *f*is the domain of the inverse *f*–1.  No horizontal line crosses the graph of *f* more than once.  The graph of the inverse is the reflection of the graph of *f* across the line *y* = *x*.    (*f*–1 ◦ *f*)(*x*) = *f*–1 (*f*(*x*)) = *x* for all *x* in the domain of *f*.  (*f ◦* *f*–1)(*x*) = *f*(*f*–1 (*x*)) = *x* for all *x* in the domain of *f*–1.  **IV. Solutions of Equations and Inequalities**  After completing this section, you should be able to:   * solve a linear equation * solve a quadratic equation * solve a rational equation * solve linear and absolute value inequalities   **A. Solution of an Equation**  For an equation of one variable such as *x*2 – 3*x* + 2 = 0, solving the equation means finding all the *x*-values which make the equation true. The numbers 1 and 2 are solutions of the equation because (1)2 – 3(1) + 2 = 0 and (2)2 – 3(2) + 2 = 0.   |  |  | | --- | --- | | The graph of *y* = *x*2 – 3*x* + 2 is shown to the right. The solutions of *x*2 – 3*x* + 2 = 0 correspond to the points where *y* is 0. Those points are the *x*-intercepts (1, 0) and (2, 0). |  | | Solving an equation of the form *f*(*x*) = 0 is the same as finding the zeros of the function *f*.  Graphically, solving an equation of the form *f*(*x*) = 0 for *real-valued* solutions is the same as finding the *x*-intercepts of the graph of *y* = *f*(*x*). |  |   There are certain basic properties of equations which are often used in finding solutions.   |  |  |  | | --- | --- | --- | | **Type of Property** | **Statement of Property** | **Example** | | Addition | If *a* = *b*, then *a* + *c* = *b* + *c*. | If *a* = *b*, then *a* + 4 = *b* + 4. | | Multiplication | If *a* = *b*, then *ac* = *bc*. | If *a* = *b*, then 5*a* = 5*b.* | | Powers | For any positive integer *n*, if *a* = *b* then *an* = *bn*. | If *a* = *b*, then *a*3 = *b*3. | | Zero products | If *ab* = 0, then *a* = 0 or *b* = 0. | If (*x* + 2)(*x* – 7) = 0,  then *x* + 2 = 0 or *x* – 7 = 0. | | Square roots | If *x*2 = *r*,  then *x* =  or *x* = –. | If *x*2 = 5, then *x* =  or *x* = –. |   **B. Linear Equations**  **Linear Equation**  A linear equation involving the variable *x* can be written in the form *ax* + *b* = 0, where *a* and *b* are constants and *a* is nonzero.  The properties of equations can be applied to solve a linear equation.   |  |  |  |  | | --- | --- | --- | --- | | **Solving the Linear Equation *ax* + *b* = 0** | | |  | | *ax* + *b* = |  |  | | *ax* + *b* + (–*b*) = | 0 + (–*b*) | Add –*b* to both sides. | | *ax* = | –*b* |  | | (1/*a*) *ax* = | (1/*a*)(–*b*) | Multiply both sides by 1/a. | | *x* = | –*b*/*a* |  | | The solution is –*b*/*a.* | | | | (–*b*/*a*, 0) is the *x*-intercept of the graph *y* = *ax* + *b*. | | | |   An equation of the form *cx* + *d* = *px* + *q* (where *c*, *d*, *p*, and *q*are constants, and *c* ≠ *p*) is also a linear equation because it can be written in the form *ax* + *b* = 0, as follows:   |  |  |  | | --- | --- | --- | | *cx* + *d* = | *px* + *q* |  | | *cx* + *d* + (–*px* – *q*) = | *px* + *q* + (–*px* – *q*) | Add (–*px* – *q*) to both sides. | | (*cx* –*px*) + (*d* – *q*) = | 0 | Regroup. | | (*c* –*p*)*x* + (*d* – *q*) = | 0 | Factor out *x.* | | *ax* + *b* = |  | Set *a* = *c* – *p* and *b* = *d* – *q.* |   The properties of equations can also be applied to solve an equation of the form *cx* + *d* = *px* + *q*.  **Example IV.B.1:** Solve 8*x* – 4 = 2*x* + 5.  **Solution:**   |  |  |  | | --- | --- | --- | | 8*x* – 4 = | 2*x* + 5 |  | | 8*x* – 4 – 2*x* = | 2*x* + 5 – 2*x* | Add –2*x*, or subtract 2*x*, to both sides. | | 6*x* – 4 = | 5 |  | | 6*x* – 4 + 4 = | 5 + 4 | Add 4 to both sides. | | 6*x* = | 9 |  | | (6*x*)/6 = | 9/6 | Multiply by 1/6, or divide by 6, on both sides. | | *x* = | 3/2 | Simplify. | | The solution is 3/2. | | |   **C. Complex Numbers**  Some equations do not have real-valued solutions but do have complex-valued solutions. Complex numbers arise from the fact that the square root of a negative number is not a real number.  **Complex Numbers**  The number *i* is defined to be the square root of –1.  That is, , and so *i*2 = –1.  A **complex number** is a number of the form *a* + *bi* where *a* and *b* are real numbers.  The **conjugate** of the complex number *a* + *bi*is *a* – *bi*.  For example,   * 5 + 2*i* is a complex number whose conjugate is 5 – 2*i*. * The conjugate of –3 – 7*i* is –3 + 7*i*. * 5*i* and –5*i* are conjugates.   Conjugates will be involved in the case where a quadratic equation has complex solutions.  **D. Quadratic Equations**  **Quadratic Equation**  A **quadratic equation** can be written in the form *ax2* + *bx* + *c* = 0 where *a*, *b*, and *c* are real numbers and *a* ≠ 0. There are three possible cases for the solutions of a quadratic equation: two real-valued solutions, one "double" real-value solution, or two complex-valued conjugate solutions.  Solving an equation of the form *ax2* + *bx* + *c* = 0 is the same as finding the *x*-intercepts of the parabola whose graph of is *y* = *ax2* + *bx* + *c*.  There are three possible cases.  Quadratic Equation  There are several approaches used to solve a quadratic equation.  **Factoring and Applying the Zero-Products Property**  If the expression *ax*2 + *bx* + *c* can be factored, then the zero-products property can be used to solve the equation *ax2* + *bx* + *c* = 0.  **Example IV.D.1:** Solve 15*x*2 – 7*x* – 2 = 0.  **Solution:**      15*x*2 – 7*x* – 2 = 0 (3*x* – 2)(5*x* + 1) = 0.  Since the product of the factors is 0,  3*x* – 2 = 0 or 5*x* + 1 = 0, so *x* = 2/3 or *x* = –1/5.  Often, a quadratic expression does not factor, and then another technique is needed. Two techniques that can be applied to any quadratic equation are described below.  **Completing the Square and Applying the Square-Roots Property**  The equation *ax2* + *bx* + *c* = 0 can always be solved by completing the square and applying the square-roots property. Recall that the technique of completing the square was reviewed in [topic I-C](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#C._Completing_the_Square).  **Example IV.D.2:** Solve 2*x*2 + 20*x* – 7 = 0.   |  |  |  | | --- | --- | --- | | **Solution:**  Begin by completing the square, and then apply the square-roots property. | | | | 2*x*2 + 20*x* – 7 = 0 |  | Divide both sides by 2, the coefficient of *x*2 term. | | *x*2 + 10*x* – 7/2 = 0 |  | Add 7/2 to both sides. | | *x*2 + 10*x* = 7/2 |  | Add [½(10)]2 = 52 = 25, to both sides. | | *x*2 + 10*x* + 25 = 7/2 + 25 |  | Factor the left side. Add the constants on the right. | |  |  | Apply the square-roots property. | |  |  | Subtract 5 from both sides. | |  |  | Rationalize the denominator. | |  |  |  | | The solutions are and . | | |   **Applying the Quadratic Formula**  The technique of completing the square can be used to derive a general rule for solving the equation *ax2* + *bx* + *c* = 0. This rule is known as the *quadratic formula*.  **Quadratic Formula:**  The quantity under the square root sign *b*2 – 4*ac*, is called the *discriminant*, designated *D*. The value of the discriminant can be used to determine the number and type of solutions.  *D* = 0: one real-number solution;  *D* > 0: two different real-number solutions;  *D* < 0: two different complex solutions that are complex conjugates.    **Example IV.D.3:** Solve 2*x*2 + 20*x* – 7 = 0.  **Solution:**  Given 2*x*2 + 20*x* – 7 = 0, note that *a* = 2, *b* = 20, and *c* = –7.  The discriminant, *D* = *b*2 – 4*ac* = 202 – 4(2)(–7) = 400 + 56 = 456 > 0, so there are two different real-number solutions. Applying the quadratic formula,    The solutions are and     **Example IV.D.4:** Solve *x*2 – 2*x* + 5 = 0.  **Solution:**  Given *x*2 – 2*x* + 5 = 0, note that *a* = 1, *b* = –2, and *c* = 5.  The discriminant, *D* = *b*2 – 4*ac* = (–2)2 – 4(1)(5) = 4 – 20 = –16 < 0, so the solutions are complex conjugates. Applying the quadratic formula,    The solutions are 1 + 2*i* and 1 – 2*i*.  **E. Rational Equations**  A **rational equation** is an equation involving a rational expression. In order to solve a rational equation, algebraically manipulate the equation to remove fractions, and solve the resulting equation. Be sure to check proposed solutions in the original equation and discard any proposed solution if it results in a zero denominator.  **Example IV.E.1:** Solve .  **Solution:**   |  |  |  |  | | --- | --- | --- | --- | |  | | Multiply both sides by *x* + 2. | | | *x*2 = 4 | |  | | | *x* = –2 or *x* = 2. | | Apply the square-roots property. | | | Check each possibility: | |  | | | For *x* = –2 |  | For *x* = 2 |  | |  |  |  |  | |  |  |  | 1 = 1 | | –2 is not a solution. | | 2 is a solution. | |   **F. Solution of Inequalities**  There are certain basic properties of inequalities which are often used finding solutions.   |  |  |  | | --- | --- | --- | | **Type of Property** | **Statement of Property** | **Example** | | Addition | If *a* < *b*, then *a* + *c* < *b* + *c*. | If *a* < *b*, then *a* + 4 < *b* + 4. | | Multiplication by a positive number | If *a* < *b* and *c* > 0, then *ac* < *bc.* | If *a* < *b*, then 5*a* < 5*b.* | | Multiplication by a negative number | If *a* < *b* and *c* < 0, then *ac* > *bc.* The direction of the inequality *reverses* when both sides are multiplied or divided by a negative number. | If *a* < *b*, then –3*a* > –3*b.* | | The properties have been stated for "<" inequalities, but analogous properties also apply for inequalities *a* ≤ *b*, *a* > *b*, and *a* ≥ *b*. | | |     **Example IV.F.1:** 5*x* + 6 < 3*x* – 2.  **Solution:**   |  |  | | --- | --- | | 5*x* + 6 < 3*x* – 2 |  | | 2*x* < –8 | Add –3*x* or subtract 3*x*, and add –6 or subtract 6. | | *x* < –4 | Multiply by ½ or divide by 2. | | The solution set is {*x* | *x* < –4} = (–∞, –4). | |   A **compound inequality** consists of inequalities joined by *and* or *or*. For example, *a* < *b* < *c* is the compound inequality *a* < *b* and *b* < *c.*  **Example IV.F.2:** 1 – 2*x* < 3 and 6*x* < 5*x* + 8.  **Solution:**   |  |  |  | | --- | --- | --- | | 1 – 2*x* < 3 | and | 6*x* < 5*x* + 8 | | –2*x* < 2 | and | *x* < 8 | | *x* > –1 | and | *x* < 8 | | –1 < *x* | and | *x* < 8 | | The solution set is {*x* | –1 < *x* < 8} = (–1, 8). | | |   **Absolute-Value Inequalities**   |  |  |  | | --- | --- | --- | | **Absolute-Value Inequality** | **Equivalent Compound Inequality** | **Solution Set in Interval Notation** | | |*x*| < *a*, for *a* > 0 | –*a* < *x* < *a* | (–*a*, *a*) | | |*x*| > *a*, for *a* > 0 | *x* < –*a* or *x* > *a* | (–∞, –*a*) union (*a*, ∞) |     **Example IV.F.3:** Solve |7 – *x*| < 5.  **Solution:**   |  |  | | --- | --- | | –5 < 7 – *x* < 5 |  | | –12 < –*x* < –2 | Subtract 7. | | 12 >*x* > 2 | Multiply by –1 and reverse direction. | | 2 < *x* < 12 | Rewrite in increasing order. | | The solution set is {*x* | 2 < *x* < 12} = (2, 12). |     **Example IV.F.4:** Solve |2*x* + 1| > 9.  **Solution:**   |  |  |  |  |  | | --- | --- | --- | --- | --- | | 2*x* + 1 | < –9 | or | 2*x* + 1 | > 9 | | 2*x* | < –10 | or | 2*x* | > 8 | | *x* | < –5 | or | *x* | > 4 | | The solution set is {*x* | *x* < –5 or *x* > 4}, or (–∞, –5) union (4, ∞), in interval notation. | | | | |   [*Return to top of page*](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/Modules/M1-Module_1/S3-Commentary.html#pagetop)  [**Report broken links or any other problems on this page.**](http://help.umuc.edu/)  [**Copyright © by University of Maryland University College.**](https://umuc.equella.ecollege.com/file/51ed41e5-be80-4110-8171-a40ed58c98af/1/MATH108-0609.zip/common/copyright.html) |